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# An extention of Girsanov type formula

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## §0. Introduction.

(A). In this note, we aim to solve the next initial value problem (0.1) by a stochastic method, which is an extention of the Girsanov type formula employed in [7]:

$$(0.1,a) \qquad \frac{\partial W}{\partial t}(t,x) = (A + B)W(t,x), \qquad t > 0,\ x \in R^d,$$

$$(0.1,b) \qquad W(0,x) = f(x).$$

Where

$$A = (-1)^{q-1} \rho \, \Sigma_{k=1}^{d} \left(\frac{\partial}{\partial x_k}\right)^{2q}$$

with a natural number $q$ and with a complex number $\rho$ such as $\mathrm{Re}\,\rho > 0$. And

$$B = \Sigma_{|\alpha| \leq 2q} \, b_\alpha(x) \, \left(\frac{\partial}{\partial x}\right)^\alpha$$

where $b_\alpha(x)$ are complex valued functions in a certain class

$\mathcal{F}^0(R^d)$, in which an initial function $f(x)$ is also included, and $\alpha = (\alpha_1,\dots,\alpha_d)$ are multi indices, denoting $|\alpha| = \Sigma_{k=1}^d \alpha_k$ and $(\partial/\partial x)^\alpha = \Pi_k (\partial/\partial x_k)^{\alpha_k}$.

Let A-process be a "Markov process" tied to

$$\frac{\partial u}{\partial t}(t,x) = A\, u(t,x), \qquad t > 0,\ x \in R^d, \tag{0.2}$$

i.e., the density of the "transition probability" for the process is taken to be the fundamental solution of (0.2).

Our policy is to study A-process as an analogue to the Brownian motion, which was proposed by Gel'fand and Yaglom [3] first. But on the way researching A-process in that direction, a great difficulty arises from the fact that its transition probability is in general a signed measure even if $\rho$ is positive ( cf. [4,5] ). Namely if a natural measure of A-process should be realized in a path space, then the measure would be of unbounded variation. This is much different from the case of the Brownian motion.

In spite of that difficulty, some mathematicians studied stochastic analysis of A-process from that point of view. When $d = 1$, Krylov [5] proved the Feynman-Kac formula for A-process, and Hochberg [4] accomplished systematic research including a definition of "stochastic integrals" of A-process. Basing on Hochberg's stochastic integrals, Berger and Sloan [1] obtained a Girsanov type formula for constant coefficients, and Motoo [6] extended that for variable coefficients in a class.

When $d \geq 2$, Berger and Sloan [2] also defined stochastic integrals by translating Hochberg's ones into the multidimensional case, and they proved Girsanov type formula for constant coefficents. But their results are partial, becasue a variety of their stochastic integrals is not sufficient if $q \geq 2$ and $d \geq 2$. In [7], we present a different definition of stochastic integrals from Berger and Sloan's, where the former includes the latter, Hochberg's, and the usual ones for the Brownian motion as special cases, respectively. Basing on those stochastic integrals, we could construct a Girsanov type formula for variable coefficients in a certain class $\mathcal{F}^{\infty}(R^d)$.

(B). Each stochastic integral in [7] has been characterized by a differential operator in $R^d$ of the order under $2q-1$. Here we shall define "singular stochastic integrals", to which the differential operators of the orders under $2q$ correspond. If such singular stochastic integrals were once established, then the Girsanov type formula, employed in [7], will enable us to solve (0.1).

We realize the above conjecture belong the following program: With §2, we begin the consideration of <u>$\varepsilon$-process</u>, that is a Markov process tied to the next parabolic equation of the order higher than (0.2):

$$(0.3) \qquad \frac{\partial u}{\partial t}(t,x) = [(-1)^{p-1} \varepsilon \, \Sigma_{k=1}^{d} (\frac{\partial}{\partial x_k})^{2p} + A]u(t,x), \qquad t > 0,\ x \in R^d,$$

where $\varepsilon$ is a positive number and $p$ is a natural number such that

$p > q$. We define the stochastic integrals of $\varepsilon$-process in a same way as in [7].

In §3, we let $\varepsilon$ tend to zero for (0.3). Then, with a suitable choice of integrands, the stochastic integrals of $\varepsilon$-proces converge to the singular stochastic integrals of A-process, while the differential operators of the orders under $2p-1$ correspond to those singular stochastic integrals. Where the sense of the convergence is taken to be a little wider than "the weak sense", proposed in [7].

The content of §4 is the Girsanov type formula. If the corresponding differential operators to the singular stochastic integrals are of the orders under $2q$, and if the coefficients $b_\alpha(x)$ for $|\alpha| = 2q$ are not "large" in comparison with $\mathrm{Re}\,\rho$, ther the Girsanov density with the singular stochastic integrals is obtained by successive approximation. We assert that our Girsanov type formula solves the "martingale problem" for $(A + B)$.

In §5, we specify the stochastic solution of (0.1), by using the above-obtained Girsanov type formula. Uniqueness and regularity of the stochastic solution are known from this specific form.

## §1. Description of notions.

Let $\mathcal{M}^\kappa(R^d)$ $\kappa \geq 0$ be the space of complex valued measures $\mu$ on $R^d$ with $\|\mu\|_\kappa \equiv \int (1 + |\xi|)^\kappa \, d|\mu|(\xi) < \infty$. $\mathcal{F}^\kappa(R^d)$ are the space of all Fourier transformations $f(x) = \int \exp\{i\langle\xi,x\rangle\}\, d\mu_f(\xi)$ of $\mu_f$ in $\mathcal{M}^{\kappa}(R^d)$, here and on $\langle\xi,x\rangle$ is the inner product in $R^d$, and we define $\|f\|_{\kappa} \equiv \|\mu_f\|_{\kappa}$. $\mathcal{M}^{\kappa}(R^d)$ are commutative Banach algebras with norms $\|\ \|_{\kappa}$ under convolution. Define $\mathcal{M}^{\infty}(R^d) \equiv \cap_{\kappa \geq 0}\mathcal{M}^{\kappa}(R^d)$ and $\mathcal{F}^{\infty}(R^d) \equiv \cap_{\kappa \geq 0}\mathcal{F}^{\kappa}(R^d)$, while the latter contains the Schwartz class $\mathcal{S}$, constants, $\sin x_k$, $\cos x_k$, and etc..

Certain "stochastic notions" of A-process and those of $\varepsilon$-process are defined in the same way as in [7]. We briefly state them. The path space $\underline{C}$ is the set of all continuous functions $w(.) = (w_1(.),\ldots,w_d(.))$: $[0,\infty) \to R^d$. We say that a function $f(w)$ on $\underline{C}$ is a tame function, if $f(w)$ is a Borel function of a finite number observations, that is

$$f(w) = f(w(t_1),\ldots,w(t_N))$$

for a Borel function $f$ on $R^{d\times N}$. Moreover if $f$ is of $\mathcal{F}^{\kappa}(R^{d\times N})$ ( respectively polynomial ), then we say that $f(w)$ is a $\mathcal{F}^{\kappa}$ ( resp. polynomial ) tame function. The Fourier transformation of the fundamental solution $p^{\varepsilon}(t,x)$ for (0.3) is

$$\exp\{-\Sigma_k(\varepsilon\ \xi_k{}^{2p} + \rho\ \xi_k{}^{2q})\ t\},$$

and $p^{\varepsilon}(t,x)$ is of the Schwarz class $\mathcal{S}$ in $x$ for each positive $t$. The expectation $E_x^{\varepsilon}[f(w)]$ of a tame function $f(w) = f(w(t_1),\ldots,w(t_N))$ $0 \leq t_1 \leq \cdots \leq t_N$ is defined by the next, if the right hand side exists:

$$(1.1)\qquad E^{\varepsilon}_{x}[f(w)] = \int\cdots\int dy^{(1)}\cdots dy^{(N)}(\Pi_n\, p^{\varepsilon}(t_n - t_{n-1}, y^{(n)} - y^{(n-1)}))$$

$$\times f(y^{(1)},\ldots,y^{(N)}),$$

where $t_0 = 0$ and $y^{(0)} = x$. We enjoy the Markov property of $\varepsilon$-process, that is: For $f$ in $\mathcal{F}^0(R^{d\times N})$, $g$ in $\mathcal{F}^0(R^{d\times N'})$, and $0 \leqq s_1 \leqq\cdots\leqq s_N \leqq t_1 \leqq\cdots\leqq t_{N'}$,

$$(1.2)\qquad E^{\varepsilon}_{x}[f(w(s_1),\ldots,w(s_N))\; g(w(t_1),\ldots,w(t_{N'}))]$$

$$= E^{\varepsilon}_{x}[f(w(s_1),\ldots,w(s_N))\; E^{\varepsilon}_{w(s_N)}[g(w(t_1),\ldots,w(t_{N'}))]].$$

We say that a sequence of tame functions $\{f_n\}$ converges in the $\varepsilon$-weak sense, if $\lim_{n\to\infty} E^{\varepsilon}_{x}[f_n\, g]$ exists for any $\mathcal{F}^{\infty}$ tame function $g$ and any $x$.

$J = (J_1,\ldots,J_d)$ is a multi index of a stochastic integral if $J_k$ $k = 1,\ldots,d$ are natural numbers such that

$$(1.3,\text{i})\qquad 2p \geqq J_k \geqq 1, \qquad k = 1,\ldots,d,$$

$$(1.3,\text{ii})\qquad |J| \equiv \Sigma_k\, J_k \geqq 2p(d-1) + 1.$$

For A-process, the expectation $E_x[f(w)]$, the Markov property, the weak sense convergence, and etc. are also defined in the same manner as just described ( see [7] ). The next simple remark states the relation of the both processes: For a $\mathcal{F}^0$ or a polynomial tame function $f(\omega)$, $\lim_{\varepsilon\to 0} E_x^{\varepsilon}[f(\omega)] = E_x[f(\omega)]$.

## §2. The stochastic integrals of $\varepsilon$-process.

(A). We fix a positive number $T$ throughout the article. For a large natural number $M$, let $\delta = T/M$, $s_m = mT/M$ $m = 0,1,\dots,M-1$, and let

$$\delta\omega_k(s_m) = \omega_k(s_{m+1}) - \omega_k(s_m), \quad k = 1,\dots,d.$$

For a multi index $J$ as in (1.3), we define

$$(2.1) \qquad (\delta\omega(s_m))^J \equiv (\tfrac{1}{\delta})^{d-1} \Pi_{k=1}^{d} (\delta\omega_k(s_m))^{J_k},$$

where we set, for $J_k = 2p$,

$$(\delta\omega_k(s_m))^{J_k} = (-1)^{p-1}\, \varepsilon\, (2p)!\, \delta.$$

<u>Theorem 2.1</u>. For $M$ and $t \in [0,T]$, let $S = S(M,T)$ be an integer such that $ST/M \leqq t < (S+1)T/M$. Let $a(x)$, $a_n(x)$ $n = 1,\dots,N$ be functions in $\mathcal{F}^\infty(R^d)$, and let $J$, $J(n)$ $n = 1,\dots,N$ be multi indices as in (1.3). Then the following sequences of tame functions converge $\varepsilon$-weakly for each $\varepsilon > 0$, as $M \to \infty$:

$$\Sigma_{m=0}^{S} a(w(s_m)) \ (\delta w(s_m))^J,$$

$$\Sigma_{m_1=N-1}^{S} \Sigma_{m_2=N-2}^{m_1-1} \cdots \Sigma_{m_N=0}^{m_{N-1}-1} \ (\Pi_{n=1}^{N} a_n(w(s_{m_n})) \ (\delta w(s_{m_n}))^{J(n)}).$$

Definition 2.2. We call the above weak limits <u>the stochastic integrals of $\varepsilon$-process</u>, and the next symbolical notations are used respectively:

$$\varepsilon\text{-}\int_0^t a(w(s)) \ (dw(s))^J,$$

$$\varepsilon\text{-}\int_0^t (dw(s_1))^{J(1)} \int_0^{s_1} (dw(s_2))^{J(2)} \cdots \int_0^{s_{N-1}} (dw(s_N))^{J(N)}$$

$$\times a_1(w(s_1)) \ a_2(w(s_2)) \cdots a_N(w(s_N)).$$

(B). We define an ordered partition $\underline{\underline{\Phi}}$, which divides the set $\{1,\ldots,d\}$ into two subsets $(\Phi_1,\Phi_2) \equiv \underline{\underline{\Phi}}$ such as:

(2.2,i) $\underline{\underline{\Phi}} = \underline{\underline{\Phi}}'$ if and only if $\Phi_r = \Phi_r'$ $r = 1,2$.

(2.2,ii) $\Phi_r$ may be empty for $r = 1$ or $2$.

The summation with respect to all such $\underline{\underline{\Phi}}$'s is denoted by $\Sigma_{\underline{\underline{\Phi}}}$.

For a multi index $J$ of a stochastic integral and an ordered partition $\underline{\underline{\Phi}}$, a constant $\chi(J,\underline{\underline{\Phi}})$ is defined by

$$
(2.3)\qquad \chi(J,\underline{\Phi}) = \begin{cases} (\Pi_{k\in\Phi_1} \dfrac{(2p)!}{(2p-J_k)!})(\Pi_{k\in\Phi_2} \dfrac{(2q)!}{(2q-J_k)!}), & \\ & \text{if } J_k \leqq 2q \text{ for } k \in \Phi_2, \\ 0, & \text{otherwise.} \end{cases}
$$

As in [7], the stochastic integrals of $\varepsilon$-process correspond to the differential operators in $R^d$.

<u>Corollary 2.3</u>. (i) Let $g$ be a $\mathcal{F}^\infty$ tame function, that is $g = g(x^{(1)},\dots,x^{(R)})$ with $x^{(r)} = w(u_r)$ $r = 1,\dots,R$ and $0 \equiv u_0 \leqq u_1 \leqq \cdots \leqq u_R \leqq T$. Then

$$
\lim_{\delta\to 0} \frac{1}{\delta} E_x^\varepsilon[\{\varepsilon\text{-}\int_t^{t+\delta} a(w(s))(dw(s))^J\}\, g(w)]
$$

$$
= \begin{cases} \Sigma_{\underline{\Phi}}\, \chi(J,\underline{\Phi})\, (-1)^{p|\Phi_1|+q|\Phi_2|+d}\, \varepsilon^{|\Phi_1|}\, \rho^{|\Phi_2|}\, E_x^\varepsilon[\, a(w(t)) & \\ \quad \times(\{\Pi_{k\in\Phi_1}(\Sigma_{r=R'}^{R} \dfrac{\partial}{\partial x_k^{(r)}})^{2p-J_k}\}\{\Pi_{k\in\Phi_2}(\Sigma_{r=R'}^{R} \dfrac{\partial}{\partial x_k^{(r)}})^{2q-J_k}\}\cdot g)], & \\ & \text{for } u_{R'-1} \leqq t < u_{R'}, \\ \Sigma_{\underline{\Phi}}\, \chi(J,\underline{\Phi})\, (-1)^{p|\Phi_1|+q|\Phi_2|+d}\, \varepsilon^{|\Phi_1|}\, \rho^{|\Phi_2|} & \\ \quad \times E_x^\varepsilon[\, a(w(t))\, (\{\Pi_{k\in\Phi_1}(\dfrac{\partial}{\partial x_k})^{2p-J_k}\}\{\Pi_{k\in\Phi_2}(\dfrac{\partial}{\partial x_k})^{2q-J_k}\}\cdot 1)\, g], & \\ & \text{for } u_R \leqq t, \end{cases}
$$

(ii) The stochastic integrals of $\varepsilon$-process are Markovian.

## §3. The singular stochastic integrals.

(A). From now on, we consider only multi indices $\alpha = (\alpha_1,\ldots,\alpha_d)$ such as $|\alpha| \leqq 2q$. For a multi index $\alpha$, we define a multi index of the stochastic integral, $J(\alpha) = (J_1(\alpha),\ldots,J_d(\alpha))$, by

$$J_k(\alpha) = 2p - \alpha_k, \qquad k = 1,\ldots,d. \tag{3.1}$$

We consider a new partiotion $\underline{\Psi}$, that is slightly different from aforesaid $\underline{\Phi}$ in (2.2). $\underline{\Psi}$ is an ordered partition which divides the set $\{1,\ldots,d\}$ into two subsets $(\Psi_1,\Psi_2) = \underline{\Psi}$ such as:

(i) $\underline{\Psi} = \underline{\Psi}'$ if and only if $\Psi_r = \Psi_r'$ $\quad r = 1,2$.

(ii) $\Psi_2$ is not empty.

The summation with respect to all such $\underline{\Psi}$'s is denoted by $\Sigma_{\underline{\Psi}}$.

For each $\alpha$ and $\underline{\Psi}$, $\Gamma(\alpha,\underline{\Psi})$ is the set of multi indices $\alpha'$ such that

$$\begin{cases} \alpha_k' = \alpha_k & \text{for } k \in \Psi_1 \\ \alpha_k' = \alpha_k + 2(p-q) & \text{for } k \in \Psi_2. \end{cases} \tag{3.2}$$

In fact, $\Gamma(\alpha,\underline{\Psi})$ consists of a element, at most.

Let measures $\nu_\alpha$, indexed by $\alpha$ for $|\alpha| \leqq 2q$, be given in $\mathcal{M}^0(R^R)$, and we define new measures $\mu_{\alpha,\beta}^{(\varepsilon)}$ of $\mathcal{M}^0(R^d)$, indexed by $\alpha$ and $\beta$, as follows:

$$
(3.3)\quad \mu^{(\varepsilon)}_{\alpha,\beta} = \begin{cases} 0 & \text{if } |\beta| \geqq |\alpha| \text{ and if } \beta \neq \alpha, \\ (\varepsilon^{d}\ \chi(J(\alpha),\underline{d}))^{-1}\ (-1)^{(p+1)d}\ \nu_{\alpha}, & \text{if } \beta = \alpha, \\ -\ \Sigma_{\underline{\Psi}}\ \Sigma_{\beta'\in\Gamma(\beta,\underline{\Psi})}[(\varepsilon^{d}\ \chi(J(\beta),\underline{d}))^{-1}\ \chi(J(\beta'),\underline{\Psi}) & \\ \qquad \times(-1)^{p(d+|\Psi_1|)+q|\Psi_2|}\ \varepsilon^{|\Psi_1|}\ \rho^{|\Psi_2|}\ \mu^{(\varepsilon)}_{\alpha,\beta'}], & \\ & \text{if } |\beta| < |\alpha|. \end{cases}
$$

Where $\chi(J(\alpha),\underline{d})$ is given by (2.3) with $\Phi_1 = \{1,\ldots,d\}$ and $\Phi_2 = \phi$. (3.3) is well defined, because (3.2) demands that

$$|\beta'| = |\beta| + 2(p - q)|\Psi_2| \geqq |\beta| + 2(p - q).$$

Let functions $b_\alpha(x)$ and $a^{(\varepsilon)}_{\alpha,\beta}(x)$ in $\mathcal{F}^0(R^d)$ be the Fourier transformations of the above-mentioned measures $\nu_\alpha$ and $\mu^{(\varepsilon)}_{\alpha,\beta}$ respectively, that is:

$$
(3.4)\quad \begin{cases} b_\alpha(x) = \int \exp\{i\langle\xi^\alpha,x\rangle\}\ d\nu_\alpha(\xi^\alpha) \\ a^{(\varepsilon)}_{\alpha,\beta}(x) = \int \exp\{i\langle\xi^\beta,x\rangle\}\ d\mu^{(\varepsilon)}_{\alpha,\beta}(\xi^\beta), \end{cases}
$$

where $\xi^\alpha$ and $\xi^\beta$, points in $R^d$, are indexed by $\alpha$ and $\beta$.

(B). Now we let $\varepsilon$ tend to zero for the stochastic integrals of $\varepsilon$-prcess.

Theorem 3.1. Let multi indices of stochastic integrals $J(\beta)$ be given by (3.1), and let $\mathcal{F}^0(R^d)$ functions $a^{(\varepsilon)}_{\alpha,\beta}(x)$ be given by (3.4). Then, for a $\mathcal{F}^0$ tame function $g(w)$, the nexts converge as $\varepsilon \to 0$:

$$\Sigma_\beta\ E^\varepsilon_x[\{\varepsilon\text{-}\int_0^t a^{(\varepsilon)}_{\alpha,\beta}(w(s))\ (dw(s))^{J(\beta)}\}\ g(w)],$$

$$\Sigma_{\beta(1)}\cdots\ \Sigma_{\beta(N)}\ E^\varepsilon_x[\{\varepsilon\text{-}\int_0^t (dw(s_1))^{J(\beta^{(1)})}\cdots\int_0^{s_{N-1}}(dw(s_N))^{J(\beta^{(N)})}$$

$$\times a^{(\varepsilon)}_{\alpha^{(1)},\beta^{(1)}}(w(s_1))\ \ldots\ a^{(\varepsilon)}_{\alpha^{(N)},\beta^{(N)}}(w(s_N))\}\ g(w)].$$

Moreover the convergences are of $\|\ \|_0$ sense for each $t$.

Definition 3.2. We call the above limits the singular stochastic integrals of A-process, which are the functionals over the space of $\mathcal{F}^0$ tame functions. Symbolically we denote them by

$$s\text{-}\int_0^t b_\alpha(w(s))\ (dw(s))^{I(\alpha)},$$

$$s\text{-}\int_0^t (dw(s_1))^{I(\alpha^{(1)})}\cdots\ \int_0^{s_{N-1}}(dw(s_N))^{I(\alpha^{(N)})}$$

$$\times b_{\alpha^{(1)}}(w(s_1))\ \ldots\times b_{\alpha^{(N)}}(w(s_N)),$$

where $I(\alpha) = (I_1(\alpha),\ldots,I_d(\alpha))$ is defined by

$$I_k(\alpha) = 2q - \alpha_k \qquad k = 1,\ldots,d.$$

(C). We state the correspondence of the singular stochastic integrals to the differential operators in $R^d$.

Corollary 3.3. (i) Let $g(w)$ be a $\mathcal{F}^\infty$ tame function, that is $g = g(x^{(1)},\ldots,x^{(R)})$ with $x^{(r)} = w(u_r)$ $r = 1,\ldots,R$ and $0 \equiv u_0 \leqq u_1 \leqq \cdots \leqq u_R \leqq T$. Then

$$\lim_{\delta\to 0} \frac{1}{\delta} E_x[\{s-\int_t^{t+\delta} b_\alpha(w(s))\ (dw(s))^{I(\alpha)}\}\ g(w)]$$

$$= \begin{cases} E_x[\ b_\alpha(w(t))\ ((\Pi_k(\Sigma_{r=R'}^R \dfrac{\partial}{\partial x_k^{(r)}})^{\alpha_k})\ g)(w)] & \text{for } u_{R'-1} \leqq t < u_{R'} \\ E_x[b_\alpha(w(t))\ ((\Pi_k(\dfrac{\partial}{\partial x_k})^{\alpha_k})\ 1)\ g(w)] & \text{for } u_R \leqq t, \end{cases}$$

(ii) The singular stochastic integrals of A-process are Markovian.

Remark 3.4. The next statement follows from Corollary 3.3, combined with Theorem 5.3 in [7]: Set $J(\alpha) = (J_1(\alpha),\ldots,J_d(\alpha))$ by $J_k(\alpha) = 2q - \alpha_k$ $k = 1,\ldots,d$, and let $b_\alpha$ be in $\mathcal{F}^\infty(R^d)$. Let $\int_0^t b_\alpha(w(s))(dw(s))^{J(\alpha)}$ be a stochastic integral defined in [7], for

A-process. If $|\alpha| \leqq 2q - 1$, then that coincides with a singular stochastic integral $s\text{-}\int_0^t b_\alpha(w(s))(dw(s))^{I(\alpha)}$ in the weak sense, except multiplication of a constant. On the other hand, the latter may exist for $|\alpha| = 2q$, while the former does not exist for the case.

Remark 3.5. For the weak existence of the singular stochastic integral, we need not assume that $|\alpha| \leqq 2q$. Thus if we replace that assumption by $|\alpha| \leqq 2p-1$, then we may define singular stochastic integrals, whose corresponding differential operators are of the orders under $2p-1$. But integrands $b_\alpha$ and a tame function $g$ should be taken in $\mathcal{F}^\infty(R^d)$ and so, now.

## §4. The Girsanov type formula with the singular stochastic integrals.

(A). We define Girsanov density with the singular stochastic integrals, in a different way from that in [7]. Our method is successive approximation, proposed in [6]. Set

$$b^* = \Sigma_{|\alpha|=2q} \|\nu_\alpha\|_0 \ ( \equiv \Sigma_{|\alpha|=2q} \|b_\alpha\|_0 ),$$

$$b^{**} = \Sigma_{|\alpha|\leqq 2q-1} \|\nu_\alpha\|_0 \ ( \equiv \Sigma_{|\alpha|\leqq 2q-1} \|b_\alpha\|_0 ).$$

Lemma 4.1. For any large number C,

$$\| E.[\{\Sigma_{\alpha^{(1)}} \cdots \Sigma_{\alpha^{(N)}}\ s\text{-}\int_0^t (dw(s_1))^{I(\alpha^{(1)})} \cdots \int_0^{s_{N-1}} (dw(s_N))^{I(\alpha^{(N)})}$$

$$\times b_{\alpha^{(1)}}(w(s_1)) \cdots b_{\alpha^{(N)}}(w(s_N))\}\ g(w(u))] \|_0$$

$$\leqq \|\mu_g\|_0 \left(\frac{b^*}{\operatorname{Re}\rho} + \frac{1}{C}\right)^N \ C^{2q} \exp\{(C\, b^{**})^{2q}\ T^{2q}\}.$$

Owing to this lemma, we may assert that:

<u>Theorem 4.2</u>. For functions $b_\alpha(x)$ in $\mathcal{F}^0(R^d)$, suppose

$$b^* \equiv \Sigma_{|\alpha|=2q} \|b_\alpha\|_0 < \operatorname{Re}\rho. \tag{4.1}$$

Then, for a $\mathcal{F}^0$ tame function $g$,

$$\Sigma_{N=0}^{\infty} \Sigma_{|\alpha^{(1)}|\leqq 2q} \cdots \Sigma_{|\alpha^{(N)}|\leqq 2q}\ E_x[\{s\text{-}\int_0^t (dw(s_1))^{I(\alpha^{(1)})} \tag{4.2}$$

$$\times \cdots \times \int_0^{s_{N-1}} (dw(s_N))^{I(\alpha^{(N)})}\ b_{\alpha^{(1)}}(w(s_1)) \cdots b_{\alpha^{(N)}}(w(s_N))\}\ g(w)]$$

converges in $\| \ \|_0$ sense.

<u>Definition 4.3</u>. We call the above weak limit <u>Girsanov</u> <u>density</u> and denote it by $Z(t,w)$, that is

$$\{(4.2)\} = E_x[Z(t,w)\ g(w)].$$

(C). We see the Markov property of $Z(t,w)$.

<u>Lemma 4.4</u>. For functions $f$ and $g$ in $\mathcal{F}^0(R^d)$,

$$E_x[Z(t,w)\ f(w(t))\ E_{w(t)}[Z(u,w)\ g(w(u))]]$$

$$= E_x[Z(t+u,w)\ f(w(t))\ g(w(t+u))].$$

Next, we shall decide the corresponding differential operator to $Z(t,w)$. Let $\mathcal{F}^{1,2q}$ be the set of functions $g(t,x)$, $(t,x) \in [0,T]\times R^d$, such that:

(4.3,i) $g(t,x) \in \mathcal{F}^{2q}(R^d)$ for each $t$, and

$$\lim_{s\to t} \|g(s,.) - g(t,.)\|_{2q} = 0.$$

(4.3,ii) For each $t$, there is a function $g_t(t,x) \in \mathcal{F}^0(R^d)$ such as

$$\lim_{s\to t} \|g_t(t,.) - \frac{g(s,.) - g(t,.)}{s - t}\|_0 = 0,$$

$$\lim_{s\to t} \|g_t(t,.) - g_t(s,.)\|_0 = 0.$$

<u>Lemma 4.5</u>. For a function $g(t,x)$ in $\mathcal{F}^{1,2q}$,

$$\lim_{u\to t} \frac{1}{u - t} E_x[Z(u,w)\ g(u,w(u)) - Z(t,w)\ g(t,w(t))]$$

$$= E_x[Z(t,w)\ ((\frac{\partial}{\partial t} + A + B)g)(t,w(t))]$$

Remark 4.6. Lemma 4.5 and a little modification of Corollary 3.4 are sufficient to state that: $Z(t,w)$ weakly solves the next stochastic differential equation,

$$Z(t,w) = 1 + \Sigma_{|\alpha| \leq 2q}\ s\text{-}\int_0^t b_\alpha(w(s))\ Z(s,w)\ (dw(s))^{I(\alpha)}.$$

(C). Here we assert that the "martingale problem" for $(A + B$ is solved by a system of the expectations $\{\tilde{E}_x[\cdot],\ x \in R^d\} \equiv \{E_x[Z(\cdot,w)\ \cdot],\ x \in R^d\}$ on the space of $\mathfrak{F}^0$ tame functions. For functions $f$ and $g$ in $\mathfrak{F}^0(R^d)$, define

$$\tilde{E}_x[(\int_0^t f(w(s))\ ds)\ g(w(u))] = \int_0^t \tilde{E}_x[f(w(s))\ g(w(u))]\ ds$$

$$( \equiv \int_0^t E_x[Z(s,w)\ f(w(s))\ g(w(u))]\ ds ).$$

This definition immediately follows from Definition 3.3, that is to define $\int_0^t f(w(s))\ ds$ by the singular stochastic integral for $|\alpha| = 0$ (see (2.1), or Remark 3.5 of this note and Remark 3.3 (i) of [7]).

Theorem 4.7. Let $f(w) = f(w(t_1),\dots,w(t_N))$ be an arbitrary $\mathfrak{F}^0$ tame function and let $t_n \leqq t \leqq u$ $n = 1,\dots,N$ for $t,\ u \in [0,T]$. Then, for any function $g$ of $\mathfrak{F}^{1,2q}$,

$$\tilde{E}_x[\{g(u,w(u)) - \int_0^u (\frac{\partial}{\partial t} + A + B)g(s,w(s))\ ds\}\ f(w)]$$

$$= \tilde{E}_x[\{g(t,w(t)) - \int_0^t (\frac{\partial}{\partial t} + A + B)g(s,w(s))\ ds\}\ f(w)].$$

## §5. The stochastic solution.

(A). Let $\mathcal{F}^{0,0}$ be the set of all functions $u(t,x)$ $(t,x) \in [0,T]\times R^d$ such as:

(5.1,i) $\quad u(t,x) \in \mathcal{F}^0(R^d)$ for each $t$.

(5.1,ii) $\quad \lim_{s\to t} \|u(t,.) - u(s,.)\|_0 = 0$ for each $t$.

<u>Definition 5.1</u>. A function $W(t,x)$ of $\mathcal{F}^{0,0}$ is <u>a wide sens</u> <u>solution</u> of (0.1), if there is a sequence of sets $\{W^{(n)}(t,x),$ $f^{(n)}(x)$, $b_\alpha^{(n)}(x)$ $|\alpha| \leqq 2q\}$ in $\mathcal{F}^{1,2q}\times\mathcal{F}^{2q}(R^d)\times\ldots\times\mathcal{F}^{2q}(R^d)$ such as:

(i) For each $n$, $W^{(n)}$ is a classical solution of (0.1) with $f = f^{(n)}$ and $b_\alpha = b_\alpha^{(n)}$ $|\alpha| \leqq 2q$.

(ii) $f^{(n)} \to f$ and $b_\alpha^{(n)} \to b_\alpha$ ( $|\alpha| \leqq 2q$ ) in $\| \|_0$ sense as $n \to \infty$, and (4.1) holds for each $b_\alpha^{(n)}$.

(iii) $\lim_n \sup_{t\in[0,T]} \|W^{(n)}(t,.) - W(t,.)\|_0 = 0.$

We define <u>a stochastic solution</u> $W(t,x)$ of (0.1) for $b_\alpha(x) = \int \exp\{i\langle\xi^\alpha,x\rangle\}\, d\nu_\alpha(\xi^\alpha)$ and $f(x) = \int \exp\{i\langle\zeta,x\rangle\}\, d\mu_f(\zeta)$ by

$$(5.2) \qquad W(t,x) \equiv E_x[Z(t,w)\, f(w(t))]$$

$$= \int d\mu_f(\zeta)\, \exp\{i\langle\zeta,x\rangle - \rho\, \Sigma_k\, \zeta_k^{2q}\, t\} +$$

$$+ \Sigma_{N=1}^{\infty} \left( \Sigma_{\alpha(1)} \cdots \Sigma_{\alpha(N)} \int d\mu_f(\zeta) \int d\nu_{\alpha(1)}(\xi^{\alpha(1)}) \cdots \int d\nu_{\alpha(N)}(\xi^{\alpha(N)}) \right.$$

$$\times \int_0^t ds_1 \cdots \int_0^{s_{N-1}} ds_N \exp\{i\langle \zeta + \xi^{\alpha(1)} + \cdots + \xi^{\alpha(N)}, x\rangle\}$$

$$\times [\Pi_{n=1}^{N} \Pi_{k=1}^{d} (i\, H_k(n))^{\alpha_k^{(n)}} \exp\{-\rho\, (H_k(n))^{2q} (s_{n-1} - s_n)\}]$$

$$\left. \times [\Pi_{k=1}^{d} \exp\{-\rho\, (H_k(N+1))^{2q} s_N\}] \right),$$

where $H_k(n) = \zeta_k + \Sigma_{r=1}^{n-1} \xi^{\alpha(r)}$ with the convention such as $\Sigma_{r=1}^{0}\{\ \} = 0$.

<u>Theorem 5.2</u>. If $b_\alpha$ and $f$ are in $\mathcal{F}^0(R^d)$, and if (4.1) holds, then (5.2) is well-defined and $W(t,x)$ is a wide sense solution of (0.1). Moreover, a solution of (0.1) is unique in the class of wide sense solutions.

<u>Remark 5.3</u>. (4.1) is a sufficient condition, under which $(A + B)$ is a strongly elliptic operator.

(B). The regularity of the stochastic solution is derived from the explicit form (5.2):

<u>Corollary 5.4</u>. Assume that (4.1) holds. If $b_\alpha$ and $f$ are in $\mathcal{F}^\kappa(R^d)$, then $W(t,x)$ is of $\mathcal{F}^\kappa(R^d)$ in $x$ for each $t$. Moreover if $\kappa = 2q$, then $W(t,x)$ is a classical solution of (0.1).

Let $\gamma$ be a number such that $0 < \gamma < 1$, and define

$$Q(\theta) \equiv \frac{1}{e(1-\gamma)}\, \theta^{1-\gamma}\, (1-\theta)^{-1} + \theta^{-\gamma}.$$

By a simple computation, we easily see that $\min_{0\leqq\theta\leqq 1} Q(\theta) \equiv Q(\theta^*) > 1$, where $0 < \theta^* < \sqrt{e}/(\sqrt{e}+1)$ is a non-negative solution of

$$\gamma\, \theta^2 - (1-\gamma)\theta - \frac{\gamma}{e(1-\gamma)}\, (1-\theta)^2 = 0.$$

<u>Corollary 5.5</u>. Let $b_\alpha(x)$'s and $f(x)$ be in $\mathcal{F}^0(R^d)$, and let $\gamma$ be a fixed number such that $0 < \gamma < 1$.

(i) Assume that

$$\Sigma_{|\alpha|=2q} \|b_\alpha\|_0\, Q(\theta^*) < \mathrm{Re}\, \rho.$$

Then $\|W(t,.)\|_{2q\gamma} < \infty$ for any $t > 0$.

(ii) If all $b_\alpha(x)$'s are of $\mathcal{F}^\delta(R^d)$ for a positive number $\delta$, and if

$$\Sigma_{|\alpha|=2q} \|b_\alpha\|_\delta\, Q(\theta^*) < \mathrm{Re}\, \rho,$$

then $\|W(t,.)\|_{2q\gamma+\delta} < \infty$ for any $t > 0$.

(iii) If the assumption of (ii) holds for $\delta \geqq 2q(1-\gamma)$, then $W(t,x)$ is a classical solution of (0.1).

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